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On the Theory of Self-Resonant Grids

By I. ANDERSON

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An approximate theory is developed to predict the frequency response of a self-resonant grid. The grid is comprised of capacitive and inductive elements and exhibits a band-stop resonance. The analysis is based upon the derivation, from physical considerations, of an equivalent circuit representation of the grid structure. Predicted results compare well with measured data.

I. INTRODUCTION

Arnaud and Pelow[1*] have recently described measurements of tbe transmission properties of several new types of self-resonant, metal grid structures. These grids, which are readily fabricated by photolithographic techniques, have applications as millimeter-wave quasi-optical filters, or diplexers, in communications satellite antennas and in beam waveguide systems. The grid elements are symmetrical such that the grids may be used with two orthogonal polarizations. In this paper, we derive theoretical expressions for the frequency response of the simplest of the new grids and compare the results with measured data.

The grid to be considered here is a periodic array of "Jerusalem" crosses as shown in Fig. 1a. We wish to determine the grid frequency response in terms of the dimensions of the elements when the planar transmitted wave is incident normally. On account of the complex geometry of the grid elements, an exact treatment as a boundary value problem would be prohibitively difficult. Computer-oriented, numerical techniques[2,3] have provided a powerful means of solution for grid structures in the form of arrays of rectangular, or circular, apertures. The successful application of these techniques requires,[4] however, considerable caution in approximating the unknown aperture fields. When the aperture geometry is complicated, as here, this aspect of the numerical approach poses a considerable difficulty.

Now, in general, the transmission properties of grid structures can be described[5] in terms of an equivalent impedance, together with a

* In eq. (2) of Ref. 1, λ should be replaced by $\lambda/2$.

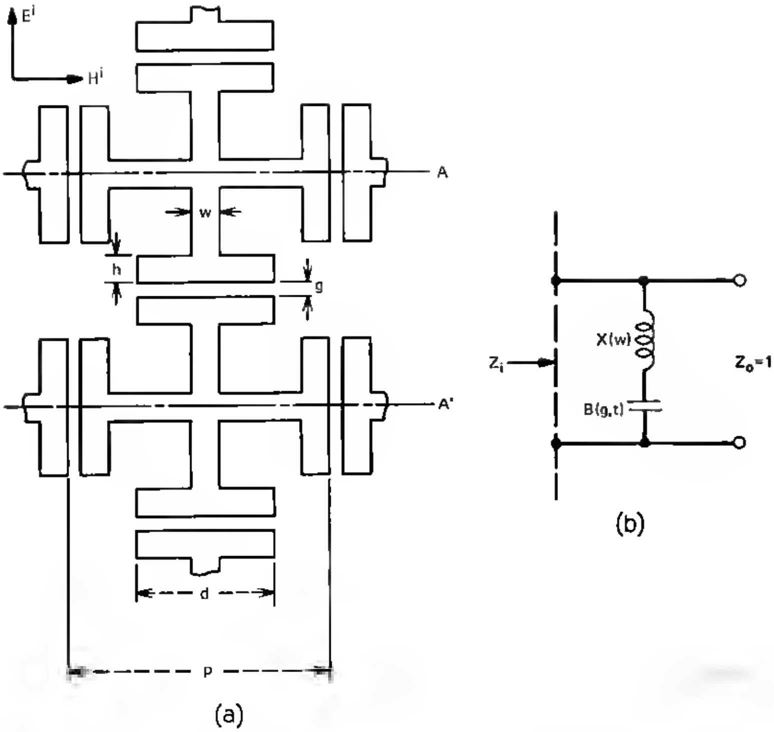

Fig. 1—Jerusalem-cross array and approximate equivalent circuit.

section of transmission line which represents propagation in free space. For example,[6] consider the transmission of a plane wave incident normally upon a grid of thin, perfectly conducting, parallel metal strips of period p. When $p \ll \lambda$, where λ is the wavelength, the equivalent impedance is a shunt inductance, or capacitance, depending upon whether the electric vector of the incident wave is parallel to, or perpendicular to, the edges of the strips. In the following section, an approximate circuit representation of the present grid is derived from physical considerations and from the known results for grids of parallel strips. This approach lends itself to a simple understanding of the grid transmission properties and, furthermore, leads to useful design formulae.

II. ANALYSIS

As shown in Fig. 1a, the period of the array is p, the width of the inductive strips is w, and the separation between adjacent crosses is g. The length and width of the capacitive segments of each cross are

d and h, respectively, and the thickness of the grid is t. It is assumed that

$$t \ll w \ll p, \qquad h \ll p < \lambda, \qquad \text{and} \qquad g \ll d \ll \lambda. \tag{1}$$

The electric field, E^i, is incident normally on the grid with the electric vector directed as shown. For purposes of discussion, we shall refer to this as the "vertical" direction; the incident magnetic field, H^i, is then in the horizontal direction. The effect of the vertical "dipoles," each of length d and width h, at the sides of the crosses is negligible for $d \ll \lambda$. It is therefore assumed that only current that flows parallel to E^i, along the vertical inductive strips and across the horizontal capacitive strips, is significant in determining the field scattered by the grid. On the basis of this assumption, we now consider the magnetic and electric fields in the vicinity of the grid.

Since $w \ll p$ and $h \ll p$, the magnetic field about the grid, due to current flowing along the vertical inductive strips, is approximately the same as that about a corresponding uniform inductive grid of period p and strip width w. Hence, the stored magnetic energy of the Jerusalem-cross grid may be represented approximately by the equivalent inductive reactance, $X(w)$, of this uniform grid, where[7]

$$X(w) = \frac{p}{\lambda}\left\{\ln\left[\operatorname{cosec}\left(\frac{\pi w}{2p}\right)\right] + F(\lambda, w)\right\} \tag{2}$$

and

$$F(\lambda, w) = \frac{Qc^2}{1 + Qs^2} + \left[\frac{pc}{4\lambda}(1 - 3s)\right]^2, \tag{3}$$

with

$$Q = \left[1 - \left(\frac{p}{\lambda}\right)^2\right]^{-\frac{1}{2}} - 1; \qquad c = \cos^2\left(\frac{\pi w}{2p}\right); \qquad s = 1 - c. \tag{4}$$

The reactance $X(w)$ is normalized with respect to the intrinsic impedance of free space. The first term in (2) can be derived[6] from magnetostatic considerations; the second term is a correction factor which is negligible when $p \ll \lambda$. Since $t \ll w$, the effect of thickness upon the inductive reactance is negligible.[8]

With regard to the distribution of electric field, it is noted, from symmetry considerations, that there is no component of electric field normal to the grid on the planes A and A' of Fig. 1a. Without disturbing the electric field we may, therefore, insert a pair of infinitely thin, perfectly conducting plates at A and A' which are perpendicular to the plane of the grid and distance p apart. In the quasi-static case, when $p \ll \lambda$, the electric flux about the grid elements within this parallel-plate transmission line is concentrated between the gaps of

the horizontal capacitive segments. We assume this concentration to be maintained at all frequencies for which $p < \lambda$. Since $g \ll d$, the effect of fringing at the extremities of the segments is negligible and the electric flux, per unit width of the parallel-plate line, is d/p times that of a corresponding uniform capacitive grid of period p, gap width g, and thickness t. This implies that the stored electric energy, of the Jerusalem-cross grid, may be represented approximately by an equivalent capacitive susceptance

$$B(g, t) = \frac{d}{p} B_u(g, t), \tag{5}$$

where $B_u(g, t)$ is the (normalized) susceptance of the corresponding uniform grid. For the case $t = 0$ we have[9]

$$B_u(g, 0) = \frac{4p}{\lambda} \left\{ \ln \left[\operatorname{cosec} \left(\frac{\pi g}{2p} \right) \right] + F(\lambda, g) \right\}, \tag{6}$$

where $F(\lambda, g)$ is given by (3) with w replaced by g. The equivalent impedance of a uniform capacitive grid of thickness t includes[6] a segment of transmission line of length t. When $t < 0.5\lambda$, this transmission line may be represented by a Π-network of shunt capacitors and a series inductor. In the present case, $t \ll \lambda$, the series element may be neglected and the total susceptance is[10]*

$$B_u(g, t) = B_u(g, 0) + \frac{2\pi p t}{\lambda g}. \tag{7}$$

The second term in (7) may be derived equivalently by considering the additional (parallel-plate) capacitance introduced by the finite thickness of a capacitive diaphragm in a parallel-plate transmission line of height p. From (5), (6), and (7), the capacitive susceptance of the Jerusalem-cross grid is approximately

$$B(g, t) = \frac{4d}{\lambda} \left\{ \ln \left[\operatorname{cosec} \left(\frac{\pi g}{2p} \right) \right] + F(\lambda, g) + \frac{\pi t}{2g} \right\}. \tag{8}$$

We have obtained approximate values of reactances with which to describe the stored magnetic and electric energies of the grid and now consider the equivalent circuit representation. It has been assumed that only current that flows vertically along the inductive strips, and across the gaps of the horizontal capacitive segments, is significant in determining the transmission properties of the grid. This suggests that the Jerusalem-cross grid can be represented approximately by a

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* In Ref. 10, the sign of the second term for B_1 in eq. (83) on p. 200 should be positive.

reactance, X_g, where

$$X_g = X(w) - \frac{1}{B(g, t)}, \tag{9}$$

shunted across a transmission line of impedance Z_o as shown in Fig. 1b. The impedances in the equivalent circuit are normalized with respect to the impedance (Z_o) of free space, and the impedance seen by a plane wave incident normally on the grid is Z_i. The grid transmission coefficient is now expressible in terms of the grid reactance X_g. The input impedance, Z_i, is

$$Z_i = \frac{jX_g}{1 + jX_g}. \tag{10}$$

The corresponding voltage reflection coefficient, R, is

$$R = \frac{Z_i - 1}{Z_i + 1} \tag{11}$$

and the grid power transmission, $|T|^2$, is

$$|T|^2 = 1 - |R|^2 = \frac{4X_g^2}{1 + 4X_g^2}. \tag{12}$$

Substituting (2) and (8) into (9) and (12) then gives the grid transmission response in terms of its dimensions. To the present order of approximation, $|T|^2$ is seen to be independent of h when $h \ll p$.

A first-order approximation for the rejection wavelength λ_r, defined by the equation $X_g = 0$, is readily found by assuming $p \ll \lambda$ so that the terms $F(\lambda, w)$ in (2) and $F(\lambda, g)$ in (8) may be neglected. Furthermore, if the effect of grid thickness is also neglected, by putting $t = 0$, and if the cosecants are replaced by the small argument forms, we find

$$X(w) \approx \frac{p}{\lambda} \ln\left(\frac{2p}{\pi w}\right), \qquad p \ll \lambda \tag{13}$$

$$B(g, t) \approx \frac{4d}{\lambda} \ln\left(\frac{2p}{\pi g}\right), \qquad p \ll \lambda,\ t = 0. \tag{14}$$

From (9), the wavelength, λ_r, at the rejection resonance is then

$$\lambda_r \approx 2\sqrt{dp \ln\left(\frac{2p}{\pi w}\right) \ln\left(\frac{2p}{\pi g}\right)}. \tag{15}$$

This result provides an approximate functional dependence of the resonant wavelength upon the grid geometry.

The effect of the dielectric sheet which supports the grid has been neglected in the preceding analysis. In general, the presence of an adjacent, low-loss, dielectric layer will increase the grid susceptance,

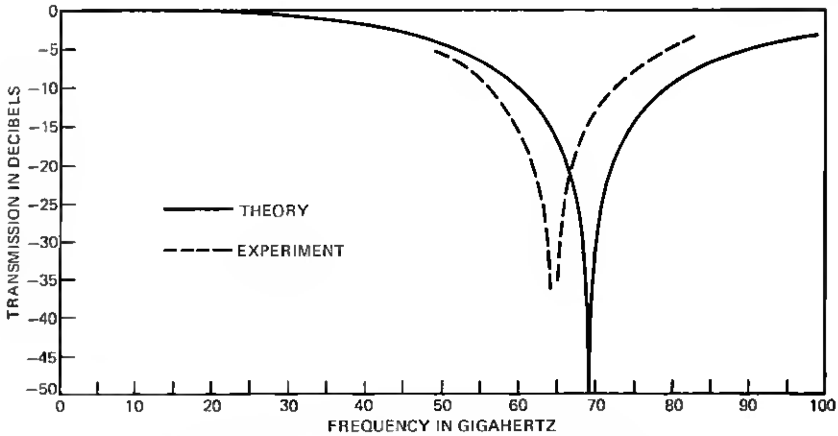

Fig. 2—Transmission response of Jerusalem-cross array.

$B(g, t)$, by modifying the electric field in the vicinity of the capacitive gaps. In the case examined by Arnaud and Pelow,[1] however, the sheet, which has a relative permittivity of about 2.5, is thinner than the grid itself and, as such, is not expected to modify the grid transmission to the present order of approximation.

III. COMPARISON WITH MEASUREMENTS

The grid measurements of Arnaud and Pelow[1] were conducted under approximately plane wave conditions and for a range of incidence angles from 5 to 45 degrees.* It was found that the frequency of the rejection resonance, and the shape of the transmission response, were practically independent of the angle of incidence within this range. Figure 2 shows the predicted frequency response for normal incidence, as obtained from (12), compared with measured data for an incidence angle of 5 degrees. The experimental curve is from Fig. 3 of Arnaud and Pelow's paper and is for a grid of dimensions p = 1.400 mm, d = 0.750 mm, $w = h$ = 0.180 mm, g = 0.090 mm, and t = 0.018 mm. The shape of the transmission response is predicted well by the theory; the error in the prediction of the rejection frequency is 7 percent. The first-order expression (15) for the rejection wavelength is within 10 percent of the value obtained from (12).

IV. CONCLUSIONS

We have examined the transmission properties of a self-resonant grid that is comprised of capacitive and inductive elements. An

* No measurements were taken at exactly normal incidence to avoid multiple reflections within the measuring system.

approximate theory has been developed to predict the frequency response of the grid when illuminated by a plane wave at normal incidence. The theory is based upon the construction of an appropriate equivalent circuit in which the values of the reactances are obtained by modification of known solutions for simple, parallel strip grids. A comparison of results with measured data shows an error of 7 percent in the prediction of the grid rejection resonance. By way of comparison, the corresponding approximate expressions (2) and (6), for parallel strip grids as obtained from rigorous analyses,[7] can be in error by about 1 to 5 percent over the range of frequencies considered here.

V. ACKNOWLEDGMENTS

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